EV-A (Advanced Course in Experimental Physics):
Lasers, Atomic Physics and Quantum Optics
Universität Erlangen–Nürnberg
Winter Term 2018/2019

Exercise Sheet 4 (15.11.2018)

Lecture: Tuesday, Thursday 12:00 - 14:00, lecture hall HH
Tutorial: Wednesday: 10:00 - 12:00, SRLP 0.179 | Thursday 08:00 - 10:00, SR 00.732, SR 01.779, SRLP 0.179
Hompage: www.qoqi.nat.fau.de/teaching/lectures

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1) Damping of electromagnetic radiation in the Abraham-Lorentz model

A usefull classical model to describe an atom is the so called Abraham-Lorentz model. According to this model the displacement $\vec{r}$ of the space-charge cloud of the electrons relative to the position of the space-charge cloud of the nucleus is defined by the following differential equation

$$m\ddot{\vec{r}} + m\gamma\dot{\vec{r}} + m\omega_0^2\vec{r} = e\vec{E}(t) \;, \tag{1}$$

where $\vec{d} = e\vec{r}$ is the dipole moment. The interesting part is the damping term $m\gamma\dot{\vec{r}}$. In this exercise it is attributed to the damping of the electromagnetic radiation - this means to the loss of energy, which an accelerated charge experiences because of emission of radiation. Remember that the radiated power of a time depended dipole moment $\vec{d}(t)$ is given by $P_{\text{rad}} = (4\pi\epsilon_0 c^3)^{-1}(2/3)\ddot{\vec{d}}^2$.
In a first step one can set up the following equation of motion:

$$\vec{F}_{\text{kons}} + \vec{F}_{\text{rad}} = m\ddot{\vec{r}} \tag{2}$$

where $\vec{F}_{\text{kons}} = e\vec{E}(t) - m\omega_0^2\vec{r}$ is the conserved force, and $\vec{F}_{\text{rad}}$ is the initially unknown recoil force - the so called Abraham-Lorentz force (which will be shown to be equal to $-m\gamma\dot{\vec{r}}$). In order to determine $\vec{F}_{\text{rad}}$ we postulate that in the time interval (t_1, t_2) the radiated energy and the work performed by $\vec{F}_{\text{rad}}$ must compensate each other

$$\int_{t_1}^{t_2} \vec{F}_{\text{rad}} \cdot \dot{\vec{r}}\mathrm{d}t \overset{!}{=} -\frac{1}{4\pi\epsilon_0}\frac{2e^2}{3c^3}\int_{t_1}^{t_2} \ddot{\vec{r}}^2\mathrm{d}t \tag{3}$$

(a) Show that for a periodic motion and a suitable choice of t_1, t_2 the postulate (3) is fulfilled if $\vec{F}_{\text{rad}}$ is chosen as

$$\vec{F}_{\text{rad}} = \frac{1}{4\pi\epsilon_0}\frac{2e^2}{3c^3}\dddot{\vec{r}}$$

With this expression for the Abraham-Lorentz force the equation of motion (2) can be rewritten as

$$\vec{F}_{\text{kons}} = m\ddot{\vec{r}} - \frac{1}{4\pi\epsilon_0}\frac{2e^2}{3c^3}\dddot{\vec{r}}$$

which is known as the *Abraham-Lorentz equation*. The Abraham-Lorentz equation is of third order in time - which implies a problem. In order to see that consider the easiest case where $\vec{F}_{\text{kons}} = 0$. Apart from the physical reasonable solution $\ddot{\vec{r}} = 0$ one also finds the non-physical solution $\ddot{\vec{r}} = \ddot{\vec{r}}(0)\mathrm{e}^{at}$. This corresponds to a increasing

acceleration - the so called *runaway*-solution.
(b) Determine the constant a.
Despite of the mentioned problem the Abraham-Lorentz equation tends to the right direction. For example, for a harmonic oscillator in the limit of small damping one can approximate $\dddot{\vec{r}} \approx -\omega_0^2 \dot{\vec{r}}$.
(c) Inserting this approximation into the Abraham-Lorentz equation show that the resulting equation of motion is identical to (1), with a microscopic expression for the damping constant given by

$$\gamma = \frac{1}{4\pi\epsilon_0} \frac{2\omega_0^2 e^2}{3mc^3}$$

Amazingly, the quantum mechanical calculation will give the same result! Therefore it is useful to have a look at the order of magnitude of the damping constant of the electromagnetic radiation.
(d) Show that γ can be expressed as

$$\gamma = \frac{2r_{\text{el}}\omega_0}{3\lambda_0}$$

with $r_{\text{el}} = \frac{1}{4\pi\epsilon_0} \frac{e^2}{mc^2}$ the *classical electron radius* and $\lambda_0 = \frac{c}{\omega_0}$ the reduced wavelength.
(e) Show that for optical transitions, $\lambda_0 \approx 10^{-7}$m, the assumption of small damping is justified. This allows the use of the non-physical Abraham-Lorentz equation in the case of a damped harmonic oscillator.

2) Electronic Dipol-Hamilton-Operator

(a) Show that the Hamiltonian $\mathcal{H} = (\vec{p} - e\vec{A})^2/(2m_e)$ describes the interaction of an electron with an electromagnetic field in the Coulomb gauge ($\vec{E} = -\partial_t \vec{A}$, $\vec{B} = \nabla \times \vec{A}$) if the electron canonical momentum is given by $\vec{p} = m_e\dot{\vec{r}} + e\vec{A}$.
(Hint: Write the Hamilton equations and demonstrate that they coincide with the equations of motion $m_e\ddot{\vec{r}} = e\vec{E} + e\vec{v} \times \vec{B}$)
(b) For the particular case of a monochromatic and spatially uniform vector field $\vec{A} = \vec{A}_0 sin(\omega t)$, demonstrate that the Hamiltonian $\mathcal{H} = (\vec{p} - e\vec{A})^2/(2m_e)$ leads to the same solution for $\vec{r}(t)$ than the (dipole) Hamiltonian $\mathcal{H} = \vec{p}^2/(2m_e) - e\vec{r} \cdot \vec{E}$.

3) Rabi oscillations

(a) Using $H = H_0 + V_{dip}$ (with $H_0 = \hbar\omega_0/2\,\hat{\sigma}_z$ and $V_{dip} = -d_{eg}E_0 \cos\omega t\,\hat{\sigma}_x$), derive by use of the Schrödinger equation $i\hbar\partial_t |\psi\rangle = H|\psi\rangle$ and the Ansatz $|\psi\rangle = \tilde{c}_g(t)|g\rangle e^{i\omega_0 t/2} + \tilde{c}_e(t)|e\rangle e^{-i\omega_0 t/2}$ in the *rotating wave approximation* (where terms oscillating at $(\omega + \omega_0)$ are neglected and only terms oscillating at $(\omega - \omega_0)$ are kept) the following coupled differential equation for the prefactors $\tilde{c}_g(t)$ and $\tilde{c}_e(t)$

$$\begin{aligned} i\dot{\tilde{c}}_g &= \frac{\Omega_R}{2} e^{i\delta t}\tilde{c}_e \\ i\dot{\tilde{c}}_e &= \frac{\Omega_R}{2} e^{-i\delta t}\tilde{c}_g. \end{aligned} \tag{4}$$

Here, the Rabi frequency Ω_R and the detuning δ are given by $\Omega_R = -d_{eg}E_0/\hbar$ and $\delta = \omega - \omega_0$, respectively.
(b) Solve the differential equation in case of resonance ($\delta = 0$) and for the initial conditions $\tilde{c}_g(t = 0) = 1$ and $\tilde{c}_e(t = 0) = 0$. Show that in this case $\tilde{c}_g(t)$ and $\tilde{c}_e(t)$ display a sinusoidal behavior. Derive the probability as a function of time that the atom is in the lower and the uppper level, respectively. Under what conditions the atom is found with 100% certainty in the upper level and under what conditions with 100% certainty in the lower level?

(c) Show by use of the differential equation (4) that for arbitrary δ one obtains the following differential equation for $\tilde{c}_e(t)$

$$\ddot{\tilde{c}}_e + i\delta\dot{\tilde{c}}_e + \left(\frac{\Omega_R}{2}\right)^2 \tilde{c}_e = 0 \tag{5}$$

(d) In order to solve (5) use the Ansatz $\tilde{c}_e(t) = Ae^{-i\alpha t}$. Show that in this case the general solution for (5) can be written in the form

$$\tilde{c}_e(t) = A_+ e^{-i\alpha_+ t} + A_- e^{-i\alpha_- t} \tag{6}$$

with $A_\pm =$ const. and $\alpha_\pm = (\delta \pm \Omega_a)/2$, whereby $\Omega_a^2 = \delta^2 + \Omega_R^2$.
(e) Show that for the initial conditions $\tilde{c}_g(t = 0) = 1$ and $\tilde{c}_e(t = 0) = 0$ one obtains from (6):

$$\rho_{ee}(t) = \left(\frac{\Omega_R}{\Omega_a}\right)^2 \sin^2\left(\frac{\Omega_a t}{2}\right)$$

(f) Sketch $\rho_{ee}(t)$ for $\delta = 0$ and $\delta = \Omega_R$. Discuss the difference of the two curves.