EV-A: Advanced Course in Experimental Physics: Lasers, Atomic Physics and Quantum Optics Universität Erlangen–Nürnberg Winter Term 2018/2019

Exercise Sheet 5 (22.11.2018)

Lecture: Tuesday, Thursday 12.00 - 14.00, lecture hall HH
Tutorials: Wednesday 10:00 - 12:00, SRLP 0.179 | Thursday 08.00 - 10.00, SR 00.732, SR 01.779, SRLP 0.179
Hompage: www.qoqi.nat.fau.de/teaching/lectures/

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1) Rabi oscillation in the H-atom

(a) Calculate the dipole matrix element d_{eg} for the $1s \to 2p$ transition in the hydrogen atom for light that is polarized along the z-direction, i.e., the matrix element $d_{eg} = \langle 1s\,|e\,z|\,2p\rangle$. The corresponding wave functions of the hydrogen atom are given by $\psi_{1s}(\mathbf{r}) = \frac{1}{\sqrt{\pi}}\left(\frac{1}{a_0}\right)^{3/2} e^{-r/a_0}$ and $\psi_{2p}(\mathbf{r}) = \frac{1}{\sqrt{\pi}}\left(\frac{1}{2a_0}\right)^{5/2} r\cos\theta e^{-r/2a_0}$, with a_0 the Bohr radius (note that for light polarized along the z-direction only transitions with $m_l = 0 \to m_l = 0$ are allowed). The level $|2p\rangle$ has a natural life time of 1,6 ns.
(b) Light, resonant with the $1s \to 2p$ transition of hydrogen at $121,6$ nm, illuminates an ensemble of monoatomic hydrogen atoms with an intensity of 10 kW/m^2. Calculate the Rabi frequency with which the atoms oscillate under these conditions and derive the time an atom needs for a full Rabi oscillation.
(c) Why is it impossible to observe Rabi oscillations even for this high intensity? How large should the intensity be in order to observe Rabi oscillations?

2) Bloch vector

In the lecture it was shown that for the state

$$\begin{aligned}|\psi\rangle &= \tilde{c}_g(t)\,|g\rangle\,e^{i\omega_0 t/2} + \tilde{c}_e(t)\,|e\rangle\,e^{-i\omega_0 t/2}\\ &= c_g(t)\,|g\rangle\,e^{i\omega t/2} + c_e(t)\,|e\rangle\,e^{-i\omega t/2}\,,\end{aligned}$$

with $c_g = \tilde{c}_g e^{-i\delta t/2}$, $c_e = \tilde{c}_e e^{i\delta t/2}$ and $\delta = \omega - \omega_0$, the mean electric dipole moment is given by

$$\langle d\rangle = \langle\psi\,|d|\,\psi\rangle = d_{eg}\left(u\cos\omega t - v\sin\omega t\right)\,,$$

where $u = 2Re(\rho_{ge}) = 2Re(c_g\,c_e^*)$ and $v = 2Im(\rho_{ge}) = 2Im(c_g\,c_e^*)$. Here, u, v and w are the components of the Bloch vector $\vec{R} = (u,v,w)$ in cartesian coordinates, with

$$w = \rho_{ee} - \rho_{gg} = |c_e|^2 - |c_g|^2\,.$$

(a) Show by using $\langle\psi|\psi\rangle = 1$ that $|\vec{R}| = 1$.
(b) Show that the prefactors c_g and c_e of the state $|\psi\rangle$ can be expressed by the polar coordinates of the Bloch vector in the following way assuming $c_g \in \mathbb{R}_+$

$$\begin{aligned}c_g &= \sin(\theta/2)\\ c_e &= e^{-i\phi}\cos(\theta/2)\end{aligned}$$

(c) Derive explicitly for $t = 0$ the prefactors c_g and c_e of the state $|\psi\rangle$ for the following points on the Bloch sphere (given in polar coordinates (θ, ϕ)):
(i) (90°, 0°),
(ii) (90°, 90°),
(iii) (90°, 180°),
(iv) (90°, -90°),
(v) (60°, 45°).

3) Rotation of the Bloch vector by a laser pulse

A pulsed Gaussian laser beam (TEM_{00} mode) is focussed to a spot of radius 1μm on a gas of atoms with a dipole moment of -10^{-29}Cm at the laser frequency ω. Similar to the case of an electric field with a constant amplitude (see last exercise), the laser pulse causes a rotation of the Bloch vector. In the case of a constant amplitude $\mathcal{E}_0$ the rotation angle calculates to $\Omega_R \cdot t = -\frac{d_{eg}\mathcal{E}_0}{\hbar} \cdot t$. Analogously, in the case of a pulsed laser, the rotation angle is given by the pulse area $\alpha = -\frac{d_{eg}}{\hbar}\int_{-\infty}^{\infty} \mathrm{d}t\, \mathcal{E}(t)$, where $\mathcal{E}(t)$ denotes the time variation of the electric field of the pulse.
a) Derive the amplitude $\mathcal{E}_0$ needed to rotate the Bloch vector by $\alpha = \pi/2$ if the time variation of $\mathcal{E}(t)$ has a Gaussian shape with a duration (FWHM) of τ_{FWHM}. Derive the required pulse energy E and calculate it for $\tau_{FWHM} = 1\,ps$ (Hint: How is the energy related to the intensity of the light field?)
b) If the atoms of the gas are initially in the ground state, find the state of the atoms at the end of the pulse.

4) Maxwell-Bloch equations

The Maxwell-Bloch equations for the electromagnetic field in a resonator $E(t)$, the polarisation $P(t) = \frac{N_{at}}{V} d_{eg}(u+iv) = \frac{N_{at}}{V} d_{eg} 2\rho_{eg}$, and the inversion density $\mathcal{N}(t) = \frac{N_{at}}{V} w(t)$ of the laser material were derived in the lecture in the following form:

$$\begin{aligned}
\partial_t E(t) &= i(\omega - \Omega + i\frac{\gamma_C}{2})E(t) + i\frac{\omega}{2\epsilon_0}P(t) \\
\partial_t P(t) &= -(\gamma' + i\delta)P(t) - i\frac{d_{eg}^2}{\hbar}\,E(t)\,\mathcal{N}(t) \\
\partial_t \mathcal{N}(t) &= -\frac{1}{\hbar}\,\mathrm{Im}\,\{P^*(t)E(t)\} - \gamma\,(\mathcal{N}(t) - \mathcal{N}_0)\,.
\end{aligned} \tag{1}$$

Here, (u, v, w) are the components of the Bloch vector, $\frac{N_{at}}{V}$ is the atomic particle density, d_{eg} the electric dipol matrix element, Ω and γ_C the eigenfrequency and the damping rate of the passive resonator, and γ' and γ are the transversal and longitudinal damping rates of the Bloch vector, respectively. ω is the angular frequency of the electromagnetic field in the resonator in presence of damping and the laser material and $\delta = \omega - \omega_0$ is the detuning of ω with respect to the atomic resonance ω_0.
Show that a new set of Maxwell-Bloch equations of the following form can be deduced from the Maxwell-Bloch equations (1) by using the expressions for the field strength in units of field strength per photon $a(t) := E(t)\sqrt{\epsilon_0 V/(\hbar\omega)}$, the total polarisation $\pi(t) := N_{at}(u+iv)$, the total inversion $n(t) := N_{at}\, w$, the Rabi frequency per photon $g := d_{eg}\sqrt{\hbar\omega/(\epsilon_0 V)}/\hbar$ and the normalised detuning $\alpha := \delta/\gamma' = (\omega - \omega_0)/\gamma'$.

$$\begin{aligned}
\partial_t a(t) &= i(\omega - \Omega + i\frac{\gamma_C}{2})a(t) + i\frac{g}{2}\pi(t) \\
\partial_t \pi(t) &= -\gamma'(1 + i\alpha)\pi(t) - i\,g\,a(t)\,n(t) \\
\partial_t n(t) &= -g\;\mathrm{Im}\,\{\pi^*(t)\,a(t)\} - \gamma\,(n(t) - n_0)\,.
\end{aligned} \tag{2}$$