EV-A (Advanced Course in Experimental Physics):
Lasers, Atomic Physics and Quantum Optics
Universität Erlangen–Nürnberg
Winter Term 2018/2019

Exercise Sheet 2 (31.10.2018)

Lecture: Tuesday, Thursday 12:00 - 14:00, lecture hall HH
Tutorial: Wednesday: 10:00 - 12:00, SRLP 0.179 | Thursday 08:00 - 10:00, SR 00.732, SR 01.779, SRLP 0.179
Hompage: www.qoqi.nat.fau.de/teaching/lectures

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1) Gaussian beams: complex source point

In order to find a solution for the paraxial wave equation (i.e., the paraxial approximation of the Helmholtz equation) one can start with a spherical wave. As shown in the lecture, in the paraxial approximation the complex amplitude of the spherical wave can be written as

$$u(\vec{r}') = \frac{1}{R(z')} e^{-i\frac{k\rho^2}{2R(z')}}$$

with the radius of curvature $R(z') = z' - z$ and distance to the optical axis $\rho^2 = (x' - x)^2 + (y' - y)^2$, where x, y, z are the coordinates of the source point of the spherical wave. The problem with this solution is that it does not drop off in ρ-direction. In order to get rid of this problem one can define a complex source point, i.e., one replaces $z \to z - iz_R$ so that one redefines $R(z') \to q(z') = z' - z + iz_R$.
(a) Rewrite $\frac{1}{q(z')}$ in the form $\frac{1}{A} - i\frac{1}{B}$ with A, $B \in \mathbb{R}$.
(b) To simplify the calculations set the source point at $x = y = z = 0$ ($z_R \neq 0$) and insert the result of (a) into $u(\vec{r}')$. Identify the new radius of curvature ($\hat{=}$ radius of the plane of equal phase at distance z). Which term ensures that the beam drops off in ρ-direction? Derive the expression for the beam waist.
(c) In the lecture the z-dependent Gouy phase $\eta(z') = \arctan(z'/z_R)$ was introduced. For the derivation, show that the following equation holds (Hint: Use the identity $\arctan(1/x) = \text{sgn}(x)\frac{\pi}{2} - \arctan(x)$)

$$\frac{iz_R}{q(z')} \equiv \frac{w_0}{w(z')} e^{i\eta(z')} .$$

2) TEM$_{00}$-Mode of a He-Ne-Laser

The TEM$_{00}$-Mode of a He-Ne-Laser ($\lambda = 632,8$nm) has a beam waist of $2\omega_0 = 1,4\,$mm.
(a) At what distance $z > 0$ from the waist is the radius of curvature minimal? What is the value for the radius of curvature at $z = 0\,$m, $z = z_R$, $z = 10\,$m and $z \to \infty$?
(b) How much does the Guoy phase of the laser vary in the range of the confocal parameter, i. e., from $z = -z_R$ to $z = +z_R$; how much does it vary from $z = -\infty$ to $z = +\infty$?
(c) How much does the beam diameter increase from $z = 0\,$m to $z = z_R$ (in units of ω_0)? How large is the beam diameter at a distance of $10\,$m?

(d) What is the divergence of the laser beam?
(e) Show that the intensity of the TEM_{00} mode for a fixed power P is given by (where $\rho^2 = x^2 + y^2$)

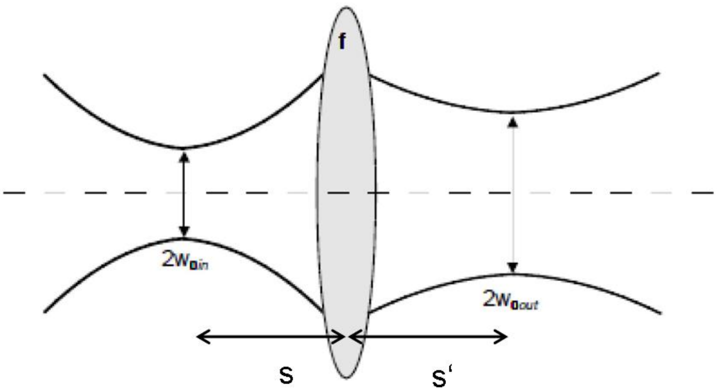
$$I_{00}(x,y,z) = \frac{2P}{\pi\omega^2(z)} e^{-2\frac{\rho^2}{\omega^2(z)}} .$$

3) Imaging a Gaussian beam (see also Applied Optics, Vol. 22, S. 658-661 (1983))

In analogy to the lens equation in geometrical optics $\frac{1}{g} + \frac{1}{b} = \frac{1}{f}$, which is valid for a thin lens of focal length f, one can use the following lens equation for imaging of a gaussian beam:

$$\frac{1}{s + z_R^2/(s-f)} + \frac{1}{s'} = \frac{1}{f}$$

Here, s is the distance of the beam waist of the incoming gaussian beam to the lens and s' the distance between the lens and the beam waist of the outgoing gaussian beam (see Fig. 1).



(a) Find an expression for s'/f and plot it as a function of s/f for $z_R/f = 1$, $z_R/f = 0,5$ and $z_R/f = 0,25$.
(b) For which value of s/f is s'/f maximal? How large is s'/f at this point?
(c) When does $s/f = s'/f$ hold? How does this value and the number of solutions depend on z_R/f? How does the solution contrast the lens equation from geometrical optics?

4) Higher-order Gaussian beams (see L. Novotny and B. Hecht, Principles of Nano-optics, p.50-52 (2006))

In the lecture Hermite-Gaussian and Laguerre-Gaussian modes were introduced by use of Hermite and generalized Laguerre polynomials. Since the paraxial Helmholtz equation is a linear homogeneous partial differential equation any combination of spatial derivatives of the TEM_{00} mode is a solution as well. It can be shown that the higher-order transversal Hermite-Gaussian and Laguerre-Gaussian modes can also be obtained from the spatial derivatives of the TEM_{00} mode using the following formulas

$$E^H_{nm}(x,y,z) = w_0^{n+m} \frac{\partial^n}{\partial x^n}\frac{\partial^n}{\partial y^m} E_{00}(x,y,z)$$

$$E^L_{nm}(x,y,z) = k^n w_0^{2n+m} e^{ikz} \frac{\partial^n}{\partial z^n}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)^m \{E_{00}(x,y,z)e^{-ikz}\}$$

a) Explicitly verify this identity for $E^H_{10}(x,y,z)$ and $E^H_{01}(x,y,z)$ and show that their superposition (with added imaginary prefactor) yields the Laguerre-Gaussian mode $E^L_{01}(\rho,\varphi,z)$. Note that: $H_1(X) = 2X$ and $L_0^{(m)}(R) = 1$
b) What does this result imply? For which kind of mirrors in the laser should one utilize the Hermite modes and for which the Laguerre modes?