EV-A: Advanced Course in Experimental Physics:
Lasers, Atomic Physics and Quantum Optics
Universität Erlangen–Nürnberg
Winter Term 2018/2019

Exercise Sheet 1 (25.10.2018)

Lecture: Tuesday, Thursday 12.00 - 14.00, lecture hall HH
Tutorials: Wednesday 10:00 - 12:00, SRLP 0.179 | Thursday 08.00 - 10.00, SR 00.732, SR 01.779, SRLP 0.179
Hompage: www.qoqi.nat.fau.de/teaching/lectures/

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1) A resonator as a periodic system of lenses

A resonator can be seen as a periodic system of lenses, that consists of two lenses with focal lengths f_1 and f_2 placed at a distance d. The complete transformation matrix σ of a light beam, that propagates through one period of the system - consisting of free propagation, lens 1, another free propagation and lens 2 - is therefore given by:

$$\sigma = L_2(f_2)P(d)L_1(f_1)P(d) = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f_2} & 1 \end{pmatrix} \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f_1} & 1 \end{pmatrix} \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}.$$

Here $L(f)$ is a matrix describing a thin lens with focal length f and $P(d)$ is a matrix corresponding to the free propagation over a distance d.

(a) Show, that a thin lens with focal length f is equivalent to two thin lenses with focal lengths $2f$, placed directly next to each other (i.e., without propagation between them):

$$L_2(f_2) = L_2(2f_2)L_2(2f_2).$$

(b) Show with the help of (a) that in the limit of many resonator roundtrips the transformation matrix can also be written as:

$$\tilde{\sigma} = L_2(2f_2)P(d)L_1(f_1)P(d)L_2(2f_2).$$

(c) Show with the help of (b) that $\tilde{\sigma}$ as a function of g_1 and g_2 is of the form:

$$\tilde{\sigma} = \begin{pmatrix} 2g_1g_2 - 1 & 2dg_1 \\ 2\frac{g_2}{d}(g_1g_2 - 1) & 2g_1g_2 - 1 \end{pmatrix}.$$

where g_1 and g_2 are given by

$$g_i = 1 - \frac{d}{R_i}, \qquad i = 1, 2.$$

(d) Calculate the eigenvalues of $\tilde{\sigma}$ and show that as a function of g_1 and g_2 they are given by:

$$\lambda_{1/2} = (2g_1g_2 - 1) \pm \sqrt{(2g_1g_2 - 1)^2 - 1}.$$

(e) Show that the eigenvectors of $\tilde{\sigma}$ are given by

$$\vec{r}_{1/2} = \alpha_{1/2} \begin{pmatrix} \pm\frac{d\sqrt{(2g_1g_2-1)^2-1}}{2g_2(g_1g_2-1)} \\ 1 \end{pmatrix},$$

with $\alpha_{1/2} \approx \tan\alpha_{1/2}$ the slope of the eigenvector ray as introduced in the lecture.
(f) Show with the help of (d) that every resonator with $0 \leq g_1g_2 \leq 1$ is stable, while every resonator with $g_1g_2 < 0$ or $g_1g_2 > 1$ is unstable.

2) Evolution of a single ray in a resonator

An arbitrary ray inside a resonator can be written in the eigenbasis as $\vec{s}_0 = \frac{c_1}{\alpha_1}\vec{r}_1 + \frac{c_2}{\alpha_2}\vec{r}_2$, where $\vec{r}_{1/2}$ are the eigenvectors given in problem 1) and $\alpha_{1/2}$ denote the slope of the eigenvector ray.
(a) Which requirements must be met by the coefficients c_1 and c_2 in order for $\vec{s}_0$ to be a physically valid ray (i.e. both entries of $\vec{s}_0 \in \mathbb{R}$).
(b) After n round trips the initial ray $\vec{s}_0$ has evolved to $\vec{s}_n = \tilde{\sigma}^n\vec{s}_0 = \frac{c_1}{\alpha_1}\lambda_1^n\vec{r}_1 + \frac{c_2}{\alpha_1}\lambda_2^n\vec{r}_2$, where $\lambda_{1/2}$ are the eigenvalues of $\tilde{\sigma}$ found in exercise 1(d). Show with the previously found conditions for c_1 and c_2 that $\vec{s}_n$ also describes a physically valid ray (i.e. both entries of $\vec{s}_n \in \mathbb{R}$).

3) Stability of resonators

Check the stability of the following resonators considering the stability diagram: